

Fragmenting D4 branes and coupled q-deformed Yang Mills.

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We compute the index of BPS states for two stacks of D4-branes wrapped on ample divisors and overlapping over a compact Riemann surface of genus $g \geq 2$ inside non-compact Calabi-Yau 3-fold. This index is given in terms of $U(N) \times U(M)$ q-deformed Yang-Mills theory with bifundamental matter. From the factorization in the limit of large D4 charge, we argue that our result computes the jump in the index of BPS states across the wall of the marginal stability for the split flow of a D4 brane fragmenting into a pair of D4 branes.

1. Introduction

The conjecture [1] of Ooguri, Strominger, Vafa relates two important enumerative problems in mathematics and physics, the counting of holomorphic curves in Calabi-Yau 3-folds and counting of BPS degeneracies of four dimensional black holes. In formulating the OSV conjecture for non-compact manifolds, one replaces BPS states of black holes with BPS states of D-branes wrapped on various cycles inside the manifold.

In a non-compact setup, OSV conjecture was successfully tested for $D4$ branes wrapped on an ample divisor $O(-p) \rightarrow \Sigma_g$ inside $O(-p) + O(p + 2g - 2) \rightarrow \Sigma_g$ [2] and for systems of $D4$ -branes wrapped on ample divisors inside toric Calabi-Yau's and intersecting over non-compact Riemann surface [3]. More recent work [4] involves both $D4$ -branes wrapped on an ample divisor and anti- $D4$ branes wrapped on a non-ample divisor and develops the connection with baby universes proposed in [5].

In the compact setup, the conjecture for black holes preserving four supercharges was tested to leading order in [6][7][8][9]. The conjecture was found to have extensions to half BPS black holes in compactifications with $\mathcal{N} = 4$ supersymmetry [10][11][6][7]. In [12] the version of the conjecture for open topological strings was formulated.

The original motivation for our work was to test OSV conjecture for the system of N $D4$ branes wrapped on ample divisor D_1 and M $D4$ branes wrapped on ample divisor D_2 intersecting over compact Riemann surface. At first we were discouraged to find that the index of BPS degeneracies Z does not have a large charge limit consistent with OSV. Instead, in this limit we obtained schematically

$$Z \sim |Z^{top}(t_N)|^2 |Z^{top}(t_M)|^2 + \dots$$

where Z^{top} is topological string partition sum and $t_N(t_M)$ indicates that Kahler modulus of Riemann surface is fixed to attractor value for $ND4$ branes wrapped on D_1 ($MD4$ branes wrapped on D_2 .) For the complete expression for Z in this limit see eq.(4.3).

However, inspired by recent results for split attractor flows [13][14], we realized that our computation should be interpreted as giving the jump in the index of BPS states across the wall of marginal stability for the split $D4 \rightarrow D4 + D4$. The Vafa-Witten theory with bifundamentals on the intersecting $D4$ branes determines the contribution to the partition function of a $D4$ wrapping the total class that arises from the corner of moduli space where the $D4$ has fragmented into these two pieces. The computation of marginal stability line for our system is similar to [14] but we include it in Section 5 to show that it can be crossed

by starting from large values of background Kähler modulus where both unfragmented and fragmented $D4$'s contribute to the BPS index.

This note is organized as follows. In Section 2 we derive the contribution of the fragmented $D4$ brane to the index of BPS degeneracies in terms of $U(N) \times U(M)$ q-deformed Yang-Mills with bifundamental matter. In section 3 we take large $D4$ charge limit and do saddle point analysis to identify configurations giving the dominant contribution to the index. In Section 4 we expand the index around this dominant contribution and observe factorization appropriate for $D4 \rightarrow D4 + D4$ split. Section 5 gives evidence for interpretation of our result as the jump in the index of BPS states across the wall of marginal stability.

2. Bound states of D4-branes intersecting over Riemann surface.

Let us consider a non-compact Calabi-Yau three-fold $O(-p+g-1) \oplus O(p+g-1) \rightarrow \Sigma_g$, that can be thought of as the local neighborhood the intersection curve of two surfaces in a compact manifold. For $p = 0$ this local model describes behavior near the canonical divisor Σ_g of the complex surface \mathcal{P} inside $O(-K) \rightarrow \mathcal{P}$. We further assume $g > 1$ and $0 \leq p < g-1$ so that both divisors $D_1 = O(-p+g-1) \rightarrow \Sigma_g$ and $D_2 = O(p+g-1) \rightarrow \Sigma_g$ have deformations.

Wrap N $D4$ branes on D_1 and M $D4$ branes on D_2 so that these two stacks of branes overlap over Σ_g . We now compute the partition function Z_g which counts (with sign) bound states of these branes. The $D0$ and $D2$ brane charges induced by bifundamental matter and fluxes on the branes are weighted by chemical potentials.

It is natural to expect that in this local geometry, the coupled four dimensional Vafa-Witten theory localizes to a 2D theory on Σ_g . This reduced theory must be q-deformed Yang Mills with gauge group $U(N) \times U(M)$ coupled to bifundamental matter. The path integral has the form

$$Z_g = \frac{1}{N!M!} \int d\Phi d\tilde{\Phi} dX_{bf} \Delta_h^{2-2g}(\Phi) \Delta_h^{2-2g}(\tilde{\Phi}) e^{-S^{(N)} - S^{(M)} - S_{bifund}} \quad (2.1)$$

where the q-YM measure was derived in [2]

$$\Delta_h(\Phi) = \prod_{1 \leq i < j \leq N} (e^{i \frac{(\Phi_i - \Phi_j)}{2}} - e^{i \frac{(\Phi_j - \Phi_i)}{2}}), \quad \Delta_h(\tilde{\Phi}) = \prod_{1 \leq a < b \leq M} (e^{i \frac{(\tilde{\Phi}_a - \tilde{\Phi}_b)}{2}} - e^{i \frac{(\tilde{\Phi}_b - \tilde{\Phi}_a)}{2}})$$

and the q-YM action is

$$S^{(N)} = \frac{\theta}{g_s} \int_{\Sigma_g} \text{Tr} \Phi \wedge \omega + \frac{1}{g_s} \int_{\Sigma_g} \text{Tr} F \Phi + \frac{g-1-p}{2g_s} \int_{\Sigma_g} \omega \wedge \text{Tr} \Phi^2 \quad (2.2)$$

$$S^{(M)} = \frac{\theta}{g_s} \int_{\Sigma_g} \text{Tr} \tilde{\Phi} \wedge \omega + \frac{1}{g_s} \int_{\Sigma_g} \text{Tr} \tilde{F} \tilde{\Phi} + \frac{g-1+p}{2g_s} \int_{\Sigma_g} \omega \wedge \text{Tr} \tilde{\Phi}^2 \quad (2.3)$$

In (2.1) dX_{bf} denotes the measure for bi-fundamentals and in (2.2),(2.3) ω stands for the unit-volume Kähler form on Σ_g .

We now specify the action for bifundamentals S_{bifund} and integrate them out in the path-integral. Our system of intersecting branes has four ND directions (these are directions normal to Σ_g in CY_3 .) Hence from flat space analysis we know that there are bifundamental fermion and bifundamental scalar. The fermion is a section of the bundle

$$S_{\Sigma} \otimes E_N \otimes E_M^* \oplus S_{\Sigma} \otimes E_N^* \otimes E_M$$

and boson is a section of the bundle

$$S_+ \otimes E_N \otimes E_M^* \oplus S_- \otimes E_N^* \otimes E_M$$

Here E_N denotes $U(N)$ vector bundle over Riemann surface Σ_g , while S_{Σ} is spin bundle over it. Finally, S_{\pm} is chiral (anti-chiral) spin bundle associated with the normal bundle to Σ_g in CY_3 . After performing a topological twist we find the topological bifundamentals: (M_1, μ_1) are 0-forms with values in $E_N \otimes E_M^*$, (h_1, v_1) are $(0, 1)$ forms with values in $E_N \otimes E_M^*$, (M_2, μ_2) are 0-forms with values in $E_N^* \otimes E_M$, (h_2, v_2) are $(0, 1)$ forms with values in $E_N^* \otimes E_M$. For $k = 1, 2$ M_k and h_k are bosons, μ_k and v_k are fermions. The BRST transformations of bifundamental fields are [15] :

$$\delta_Q M_k = \mu_k, \quad \delta_Q \mu_k = -i(\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi}) M_k, \quad k = 1, 2$$

$$\delta_Q v_k = h_k, \quad \delta_Q h_k = -i(\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi}) v_k, \quad k = 1, 2.$$

Recall also BRST transformations of the basic multiplet of cohomological 2D Yang Mills theory [15] :

$$\delta_Q A_{\mu} = i\psi_{\mu}, \quad \delta_Q \psi_{\mu} = -D_{\mu} \Phi, \quad \delta_Q \Phi = 0$$

$$\delta_Q \tilde{A}_{\mu} = i\tilde{\psi}_{\mu}, \quad \delta_Q \tilde{\psi}_{\mu} = -D_{\mu} \tilde{\Phi}, \quad \delta_Q \tilde{\Phi} = 0$$

We consider the Q-exact action for bifundamentals (we skip writing appropriate trace in the formulae below):

$$S_{bifund} = S_{bifund}^{(1)} + S_{bifund}^{(2)}, \quad S_{bifund}^{(k)} = \{Q, V_k + tV'_k\} \quad k = 1, 2$$

$$V_k = \int_{\Sigma_g} \left(\bar{v}_k D_{\bar{z}} M_k + D_z (\bar{M}_k) v_k \right), \quad V'_k = t \int_{\Sigma} \left(e \bar{\mu}_k M_k + \bar{v}_k h_k \right), \quad k = 1, 2.$$

where $e = \sqrt{g}$. Since S_{bifund} is Q-exact the partition function is independent of the parameter t , hence we are free to take it large to render the bifundamental fields heavy. Below we integrate out bifundamentals M_1, μ_1, v_1, h_1 and at the end comment on the result for integrating out fields M_2, μ_2, v_2, h_2 .

The action $S_{bifund}^{(1)}$ has the form:

$$\begin{aligned} S_{bifund}^{(1)} = \int_{\Sigma} & \left(t \bar{h}_1 h_1 + \bar{h}_1 D_{\bar{z}} M_1 + D_z \bar{M}_1 h_1 + t e \bar{\mu}_1 \mu_1 + t e \bar{M}_1 (\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi}) M_1 \right. \\ & \left. + t \bar{v}_1 (\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi}) v_1 - \bar{v}_1 (\psi_{\bar{z}} - \tilde{\psi}_{\bar{z}}) M_1 + \bar{M}_1 (\psi_z - \tilde{\psi}_z) v_1 - \bar{v}_1 D_{\bar{z}} \mu_1 + D_z \bar{\mu}_1 v_1 \right) \end{aligned}$$

Integrating out h_1 and changing variable

$$w_1 = v_1 - \frac{1}{t} (\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi})^{-1} (D_{\bar{z}} \mu_1 + (\psi_{\bar{z}} - \tilde{\psi}_{\bar{z}}) M_1)$$

we find the remaining action

$$\begin{aligned} S_{bifund}^{(1)} = t \int_{\Sigma} & \left[e \left(\bar{\mu}_1 \mu_1 + \bar{M}_1 (\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi}) M_1 \right) + \bar{w}_1 (\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi}) w_1 \right] \\ & - \frac{1}{t} \int_{\Sigma} \left[D_z \bar{M}_1 D_{\bar{z}} M_1 + \left(D_z \bar{\mu}_1 + \bar{M}_1 (\psi_z - \tilde{\psi}_z) \right) (\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi})^{-1} (D_{\bar{z}} \mu_1 + (\psi_{\bar{z}} - \tilde{\psi}_{\bar{z}}) M_1) \right] \end{aligned}$$

In the limit of large t we can drop the terms of order $O(\frac{1}{t})$ in the action. Let us introduce a basis of eigenmodes of $D_z D_{\bar{z}}$:

$$D_z D_{\bar{z}} f^I = \lambda_I \epsilon_{z\bar{z}} f^I$$

Then we expand

$$M_1 = \sum_I M_{1(I)} f^I, \quad \mu_1 = \sum_I \mu_{1(I)} f^I.$$

For non-zero eigenvalue $\lambda_I \neq 0$ we also define $g_{\bar{z}}^I = \frac{1}{\sqrt{\lambda_I}} D_{\bar{z}} f^I$ and expand w_1 as

$$w_{1\bar{z}} = \sum_{I: \lambda_I \neq 0} w_{1(I)} g_{\bar{z}}^I + \sum_{a=1}^{h^1} w_{1a} g_{\bar{z}}^a$$

Here $h^m = \dim H^m(\Sigma, E_N \otimes E_M^*)$ and $g_{\bar{z}}^a$ for $a = 1, \dots, h^1$ are zero modes. Altogether, we find that integrating out bifundamentals M_1, μ_1, v_1, h_1 contributes to the measure for Φ and $\tilde{\Phi}$:

$$\det^{h^1 - h^0}(\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi})$$

Analogously, integrating out bifundamentals M_2, μ_2, v_2, h_2 contributes to the measure for Φ and $\tilde{\Phi}$:

$$\det^{k^1 - k^0}(\Phi \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{\Phi})$$

where we denote $k^m = \dim H^m(\Sigma_g, E_N^* \otimes E_M)$. Now we use Riemann-Roch theorem to compute

$$h^1 - h^0 = g - 1 + c_1(E_M) - c_1(E_N), \quad k^1 - k^0 = g - 1 + c_1(E_N) - c_1(E_M)$$

where $c_1(E_N)$ is the first Chern class of bundle E_N . We conclude that integrating out all bifundamentals contributes to the measure

$$\Sigma^{2(g-1)}(\Phi, \tilde{\Phi}) \tag{2.4}$$

where

$$\Sigma(\Phi, \tilde{\Phi}) = \prod_{i=1}^N \prod_{b=1}^M \left(e^{i \frac{(\Phi_i - \tilde{\Phi}_b)}{2}} - e^{i \frac{(\tilde{\Phi}_b - \Phi_i)}{2}} \right)$$

and we took into account that $\Phi, \tilde{\Phi}$ are periodic.

Hence the total measure in the resulting path-integral over $\Phi, \tilde{\Phi}$ has the form

$$\mathcal{G}(\Phi, \tilde{\Phi}) = \frac{\Delta_h^{2-2g}(\Phi) \Delta_h^{2-2g}(\tilde{\Phi})}{\Sigma^{2(1-g)}(\Phi, \tilde{\Phi})} \tag{2.5}$$

and the path integral (2.1) is brought to the form

$$Z_g = \frac{1}{N!M!} \int d\Phi d\tilde{\Phi} \mathcal{G}(\Phi, \tilde{\Phi}) e^{-S^{(N)} - S^{(M)}} \tag{2.6}$$

After summing over all flux configurations

$$F_i = 2\pi r_i \omega, \quad \tilde{F}_a = 2\pi s_a \omega, \quad r_i, s_a \in \mathbf{Z} \quad i = 1, \dots, N, \quad a = 1, \dots, M$$

path integral localizes to

$$\Phi_i = ig_s n_i, \quad \tilde{\Phi}_b = ig_s m_b$$

and we find

$$Z_g = \frac{1}{N!M!} \sum_{n_i, m_a \in \mathbf{Z}} \mathcal{G}(ig_s \vec{n}, ig_s \vec{m}) q^{-\frac{g-1-p}{2} \vec{n}^2 - \frac{g-1+p}{2} \vec{m}^2} e^{i\theta \left(\sum_{i=1}^N n_i + \sum_{a=1}^M m_a \right)} \quad (2.7)$$

where $q = e^{-g_s}$. Let us rewrite partition function Z_g as a sum over representations \mathcal{R} of $U(N)$ and \mathcal{Q} of $U(M)$.

$$Z_g^{(N,M)} = \Upsilon \sum_{\mathcal{R}-U(N), \mathcal{Q}-U(M)} (Z_{\mathcal{R}\mathcal{Q}})^{2-2g} \tilde{q}^{\frac{g-1-p}{2} C_2(\mathcal{R}) + \frac{g-1+p}{2} C_2(\mathcal{Q})} e^{i\theta (C_1(\mathcal{R}) + C_1(\mathcal{Q}))} \quad (2.8)$$

where $\tilde{q} = q^{-1}$ is small expansion parameter and

$$Z_{\mathcal{R}\mathcal{Q}} = \tilde{q}^{\frac{NC_1(\mathcal{Q}) + MC_1(\mathcal{R})}{2}} \sum_{A-SU(M)} S_{A\mathcal{Q}}^{(M)}(\tilde{q}) S_{A\mathcal{R}}^{(N)}(\tilde{q}) \quad (2.9)$$

and sum goes over Young diagrams A with number of rows less than M . We have assumed $N \geq M$ and used

$$\Sigma^{-1}(ig_s \vec{n}, ig_s \vec{m}) = \tilde{q}^{\frac{NC_1(\mathcal{Q}) + MC_1(\mathcal{R})}{2}} \sum_{A-SU(M)} Tr_A(\tilde{q}^{\mathcal{Q} + \rho^{(M)}}) Tr_A(\tilde{q}^{\mathcal{R} + \rho^{(N)}})$$

as well as the definition of S-matrix

$$S_{A\mathcal{Q}}(q) = Tr_{\mathcal{Q}}(q^{\rho^{(M)}}) Tr_A(q^{\mathcal{Q} + \rho^{(M)}})$$

where $\rho_i^{(N)} = \frac{N-2i+1}{2}$ is the Weyl vector of $U(N)$. The overall normalization, Υ , is ambiguous and can be fixed from the requirement of factorization of Z_g in the large N, M limit similar to [2].

2.1. Cap in the holonomy basis

One can think of the partition function (2.8) as obtained from an operatorial approach, i.e. by sewing $2g - 2$ pants. Then the cap in the holonomy basis is given by

$$C(U, V) = \sum_{\mathcal{R}, \mathcal{Q}} Tr_{\mathcal{R}} U Tr_{\mathcal{Q}} V Z_{\mathcal{R}\mathcal{Q}} = \quad (2.10)$$

$$\frac{1}{\Delta_h(u)\Delta_h(v)} \sum_{\mu} \sum_{w \in S_M} \sum_{\sigma \in S_N} (-)^{w+\sigma} \delta\left(v + i\frac{g_s N}{2} + ig_s w(\mu + \rho^{(M)})\right) \delta\left(u + i\frac{g_s M}{2} + ig_s \sigma(\mu + \rho^{(N)})\right)$$

where $U = e^{iu}$, $V = e^{iv}$.

Recall that, as discussed in [3], insertion of the operator $Tr_{\mathcal{R}} e^{i\Phi}$ into the path-integral of the $U(N)$ q-YM on the cap gives

$$Z_N(C, Tr_{\mathcal{R}} e^{i\Phi}) = \frac{1}{\Delta_h(u)} \sum_{\sigma \in S_N} (-)^{\sigma} \delta\left(u + ig_s \sigma(\mathcal{R} + \rho^{(N)})\right) \quad (2.11)$$

So that the cap $C(U, V)$ is simply

$$C(U, V) = \sum_{\mu-SU(M)} Z_N(C, Tr_{\mu} e^{i\Phi}) Z_M(C, Tr_{\mu} e^{i\tilde{\Phi}})$$

We can motivate this by consistency as in [3], assuming for simplicity that $N = M = 1$.

The operator insertion (2.11) enforces the delta function in (2.10) which, in the case $N = M = 1$, simply says that

$$u = -v.$$

Recall that the flux of the Vafa-Witten gauge field living on each of the D4 branes is computed in the two dimensional reduction by the holonomy of the gauge field around the boundary of the cap,

$$u = \int_{Cap} F^{(1)} = - \int_{Cap} F^{(2)} = -v.$$

Therefore, along the intersection cap, the fluxes must be the same.

It is suggestive that in the mathematical description of the classical theory, namely the moduli space of sheaves with support on the intersecting pair of divisors, a similar condition arises. In particular, stable coherent sheaves supported on the union of D_1 and D_2 are defined by a sheaf on D_1 and one on D_2 , together with an isomorphism from $\mathcal{E}_1|_{\Sigma_g} \rightarrow \mathcal{E}_2|_{\Sigma_g}$, as shown in section 3 of [16]. In the rank 1 case, this isomorphism serves to identify the fluxes along Σ_g , precisely as we find in the cap amplitude. The two dimensional theory we have constructed should thus be regarded as the correct quantization of the classical description of this moduli space of sheaves.

Note that the quantization in units of g_s that is also encoded by (2.10) is automatic in the q-deformed Yang-Mills theory, thus aside from the positivity of the row lengths of μ , the insertion (2.11) is implied by the natural $U(N) \times U(M)$ covariant generalization of the requirement of equal flux along the intersection curve.

3. Saddle point analysis.

Suppose that the genus, $g \geq 2$, and that $\theta = 0$. Then we will determine the dominant contribution to the partition function (2.7) in the large N and M limit.

Let us first set $p = 0$. The term $\frac{1}{\Delta_h(n)^{2g-2}}$ results in an attractive force between the N eigenvalues, which are also pushed to the bottom of the quadratic potential, $g_s(g-1)n^2/2$. The same holds for the M eigenvalues. The effect of the opposite statistics of the bifundamental multiplets is that the n_i repel the m_j . Recalling that all the $N + M$ eigenvalues must be distinct, we see that the dominant contribution consists of a clump of N and a clump of M , either touching or separated by some distance. These two possibilities correspond, respectively, to the existence of two or four Fermi seas.

Let us parameterize the position of the clumps by their midpoints at position x for the N and y for the M . Clearly, the attractive interaction among the n_i is unaffected by variation of x , and likewise for the M eigenvalues. The repulsive force is given by $(g-1)g_s \coth\left(\frac{g_s}{2}(n_i - m_j)\right)$ between two eigenvalues of opposite type. This is strictly stronger than the constant force of $g_s(g-1)$, which is approached in the limit $n_i - m_j \gg 0$.

We will use this approximation $[n - m]_q \sim e^{|g_s(n-m)|/2}$, hence the actual separation is at least as great as what we find here. The action of the clump of N depends on x in the large N limit as

$$\frac{g_s(g-1)}{2} \sum_{i=-N/2}^{N/2} (x+i)^2 \approx \frac{g_s(g-1)}{6} \left(\left(x + \frac{N}{2}\right)^3 - \left(x - \frac{N}{2}\right)^3 \right) = g_s(g-1)x^2 N/2,$$

up to terms independent of x or subleading in $1/N$. Putting this together with the constant repulsive force between the clumps (which gives a total of $NMg_s(g-1)$), we find the saddle point conditions

$$g_s(g-1)Nx = g_s(g-1)MN$$

$$g_s(g-1)My = -g_s(g-1)MN.$$

This has solution $x = M$ and $-y = N$, which has two clumps whose centers are separated by distance $N + M$.

For $0 < p < g-1$ we find analogously:

$$x_p = \frac{(g-1)M}{g-1-p}, \quad y_p = -\frac{(g-1)N}{g-1+p}$$

Please note that working in the regime $q^{-1} \ll 1$ is crucial for the existence of the saddle point. There is no saddle point in the other regime $q \ll 1$.

4. Large N and M factorization for $g \geq 2$.

To take the large N and M limit of the partition sum Z_g we use (2.7). Given the picture we just determined of the dominant contribution to the partition function (2.7), it is natural to parametrize the eigenvalues as

$$\vec{n} = x_p + \rho^{(N)} + R_+ \overline{R}_-[l_R], \quad \vec{m} = y_p - \rho^{(M)} - Q_+ \overline{Q}_-[l_Q]$$

where $U(N), U(M)$ representations are written in terms of coupled representations

$$\mathcal{R} = R_+ \overline{R}_-[l_R], \quad \mathcal{Q} = Q_+ \overline{Q}_-[l_Q]$$

For simplicity we assume that N and M are even.

To make contact with OSV conjecture, we do analytic continuation and consider q as a small expansion parameter below. Then we have

$$\frac{\Sigma(ig_s \vec{n}, ig_s \vec{m})}{\Delta_h(ig_s \vec{n}) \Delta_h(ig_s \vec{m})} = \frac{q^{-\frac{NM(x_p - y_p)}{2}} q^{-\frac{NM}{2}(l_R + l_Q)} q^{-\frac{N}{2}(|Q_+| - |Q_-|)} q^{-\frac{M}{2}(|R_+| - |R_-|)}}{S_{0[R_+ \overline{R}_-]} S_{0[Q_+ \overline{Q}_-]}}$$

$$\prod_{i=1}^{N/2} \prod_{j=1}^{M/2} (1 - q^{x_p - y_p + l_R + l_Q} q^{(\rho_N^+ + R_+)_i + (\rho_M^+ + Q_+)_j}) (1 - q^{x_p - y_p + l_R + l_Q} q^{(\rho_N^+ + R_+)_i - (\rho_M^+ + Q_-)_j})$$

$$(1 - q^{x_p - y_p + l_R + l_Q} q^{-(\rho_N^+ + R_-)_i + (\rho_M^+ + Q_+)_j}) (1 - q^{x_p - y_p + l_R + l_Q} q^{-(\rho_N^+ + R_-)_i - (\rho_M^+ + Q_-)_j}),$$

where $\rho_N^+ = (\frac{N-1}{2}, \frac{N-3}{2}, \dots, \frac{1}{2})$ is the positive half of the Weyl vector, ρ_N .

Recalling the Schur function identity,

$$\sum_{\eta} s_{\eta}(x) s_{\eta^t}(y) = \prod_{i,j} (1 + x_i y_j),$$

we see that each of the interaction terms between the four Fermi surfaces can be expanded in the form

$$\prod_{i,j} (1 - q^{x_p - y_p + l_R + l_Q} q^{(\rho_N^+ + R_+)_i + (\rho_M^+ + Q_+)_j}) = \sum_{\eta} s_{\eta}(q^{\rho_N^+ + R_+}) (-)^{|\eta|} q^{(x_p - y_p + l_R + l_Q)|\eta|} s_{\eta^t}(q^{\rho_M^+ + Q_+}).$$

It is crucial that η is a Young diagram, not a $U(N)$ representation, hence it has positive rows lengths, and must be small due to the suppression by $q^{(x_p - y_p)|\eta|}$. This implies that $s_{\eta}(q^{\rho_N^+ + R_+}) \rightarrow q^{N|\eta|/2} \frac{W_{\eta R_+}(q)}{W_{R_+}(q)}$ in the large N, M limit. For convenience, let the distance between the clumps be denoted by $d_p = x_p - y_p = \frac{g-1}{(g-1)^2 - p^2} ((g-1)(N+M) - p(N-M))$.

Therefore, using the factorization formula for $S_{0[R_+ \overline{R}_-]}$ derived in [2], we find that in the large N and M limit:

$$\begin{aligned} \Sigma(ig_s \vec{n}, ig_s \vec{m}) &= q^{-NMd_p/2} q^{-\frac{NM}{2}(\ell_R + \ell_Q)} q^{-\frac{N}{2}(|Q_+| - |Q_-|)} q^{-\frac{M}{2}(|R_+| - |R_-|)} \\ &\sum_{\eta_{ab}} (-) \sum |\eta_{ab}| e^{-\sum_{a,b} ((t_a - g_s \ell_R) + (\tilde{t}_b - g_s \ell_Q)) |\eta_{ab}|} \prod_{a,b} \frac{W_{\eta_{ab} R_a}(q^a)}{W_{R_a}(q^a)} \frac{W_{\eta_{ab}^t Q_b}(q^b)}{W_{Q_b}(q^b)}, \end{aligned}$$

where $a, b = \pm$ parametrize the couplings between pairs of Fermi seas, the W-functions are evaluated at either q or q^{-1} as indicated, and we have defined

$$\begin{aligned} t_a &= g_s \frac{(g-1)N}{g-1+p} + (-)^a g_s \frac{N}{2} \\ \tilde{t}_b &= g_s \frac{(g-1)M}{g-1-p} + (-)^b g_s \frac{M}{2}. \end{aligned}$$

The other piece in Z_g in the large N, M limit has the form

$$\begin{aligned} &q^{-\frac{1}{2} \left(x_p^2(g-1-p) + y_p^2(g-1+p) \right)} q^{-M(g-1)(Nl_R + |R_+| - |R_-|)} q^{-N(g-1)(Ml_Q + |Q_+| - |Q_-|)} \\ &\times q^{-\frac{(g-1-p)}{2} C_2 \left(R_+ \overline{R}_- [l_R] \right)} q^{-\frac{(g-1+p)}{2} C_2 \left(Q_+ \overline{Q}_- [l_Q] \right)} \end{aligned}$$

Using expressions for quadratic Casimirs of $R_+ \overline{R}_- [l_R]$ and $Q_+ \overline{Q}_- [l_Q]$ (see for example [2]) we find

$$Z_g \sim |Z_{top}(t)|^2 |Z_{top}(\tilde{t})|^2 + \dots \quad (4.1)$$

where the Kahler modulus in the first (second) Z_{top} factor is attractor value for N D4 branes wrapped on D_1 (M D4 branes wrapped on D_2).

$$t = \frac{(g-1+p)N}{2} + i\theta_1, \quad \tilde{t} = \frac{(g-1-p)M}{2} + i\theta_2 \quad (4.2)$$

Fixing the normalization factor by requiring the partition function to factorize, we need $\Upsilon = \alpha(g_s, \theta; N) \alpha(g_s, \theta; M) q^{NM(g-1)d_p/2}$, where α is defined in [2]; the classical pieces of the prepotential we find will be identical to the \hat{Z}_0 of [2]. Putting everything together, we have the full factorization formula:

$$\begin{aligned} Z &= \sum_{P_i \tilde{P}_i \eta_{ab}^i} Z_N^+(g_s; t - (g-1-p)g_s \ell_R, t^+ - g_s \ell_R) Z_N^-(g_s; \bar{t} + (g-1-p)g_s \ell_R, t^- - g_s \ell_R) \\ &\quad Z_M^+(g_s; \tilde{t} - (g-1+p)g_s \ell_Q, \tilde{t}^+ - g_s \ell_Q) Z_M^-(g_s; \bar{\tilde{t}} + (g-1+p)g_s \ell_Q, \tilde{t}^- - g_s \ell_Q), \end{aligned}$$

where the chiral blocks are defined by

$$\begin{aligned}
Z_N^+(g_s; t, t^+) &= \hat{Z}_0(t) e^{-\frac{t}{g-1+p}} \sum |P_i| e^{-t^+} \sum |\eta_{+b}^i| \sum_R \frac{q^{\frac{1}{2}(g-1+p)\kappa_R} e^{-t|R|}}{W_R^{2g-2}} \prod_{i=1}^{2g-2} \left(\frac{W_{P_i R}}{W_R} \frac{W_{\eta_{++}^i R}}{W_R} \frac{W_{\eta_{+-}^i R}}{W_R} \right) \\
Z_N^-(g_s; t, t^-) &= \hat{Z}_0(t) e^{-\frac{t}{g-1+p}} \sum |P_i| e^{-t^-} \sum |\eta_{-b}^i| \sum_R \frac{q^{\frac{1}{2}(g-1+p)\kappa_R} e^{-t|R|}}{W_R^{2g-2}} \prod_{i=1}^{2g-2} \left(\frac{W_{P_i R}}{W_R} \frac{W_{\eta_{-+}^i R^t}}{W_{R^t}} \frac{W_{\eta_{--}^i R^t}}{W_{R^t}} \right) \\
Z_M^+(g_s; \tilde{t}, \tilde{t}^+) &= \hat{Z}_0(\tilde{t}) e^{-\frac{\tilde{t}}{g-1-p}} \sum |\tilde{P}_i| e^{-\tilde{t}^+} \sum |\eta_{a+}^i| \sum_R \frac{q^{\frac{1}{2}(g-1-p)\kappa_R} e^{-\tilde{t}|R|}}{W_R^{2g-2}} \prod_{i=1}^{2g-2} \left(\frac{W_{\tilde{P}_i R}}{W_R} \frac{W_{\eta_{++}^i R}}{W_R} \frac{W_{\eta_{+-}^i R}}{W_R} \right) \\
Z_M^-(g_s; \tilde{t}, \tilde{t}^-) &= \hat{Z}_0(\tilde{t}) e^{-\frac{\tilde{t}}{g-1-p}} \sum |\tilde{P}_i| e^{-\tilde{t}^-} \sum |\eta_{a-}^i| \sum_R \frac{q^{\frac{1}{2}(g-1-p)\kappa_R} e^{-\tilde{t}|R|}}{W_R^{2g-2}} \prod_{i=1}^{2g-2} \left(\frac{W_{\tilde{P}_i R}}{W_R} \frac{W_{\eta_{+-}^i R^t}}{W_{R^t}} \frac{W_{\eta_{--}^i R^t}}{W_{R^t}} \right).
\end{aligned}$$

This can be expressed in a more suggestive form by trading the sums over “ghost” representations for integrals over $SU(\infty)$ matrixes associated to the noncompact moduli. In particular, we have:

$$\begin{aligned}
Z &= \sum_{\ell_R, \ell_Q} \int d\vec{U} d\vec{V} d\vec{W}_{\pm\pm} \psi(g_s; t - (g-1-p)g_s \ell_R, \vec{U}, \vec{W}_{+\pm}) \bar{\psi}(g_s; t + (g-1-p)g_s \ell_R, \vec{U}, \vec{W}_{-\pm}) \\
&\quad \psi(g_s; \tilde{t} - (g-1+p)g_s \ell_Q, \vec{V}, \vec{W}_{\pm+}) \bar{\psi}(g_s; \tilde{t} + (g-1+p)g_s \ell_Q, \vec{V}, \vec{W}_{\pm-}),
\end{aligned} \tag{4.3}$$

where the contour integrals are over matrixes of the form $U = e^u$ for u with fixed real part given by the attractor value of the noncompact moduli:

$$Re[u] = \frac{2t}{g-1+p}, \quad Re[v] = \frac{2\tilde{t}}{g-1-p}, \quad Re[w_{ab}] = t^a + \tilde{t}^b.$$

5. Fragmenting 4-branes

We conjecture that $U(N) \times U(M)$ q-deformed Yang-Mills theory with bifundamental matter is the microscopic worldvolume description of those states in the BPS Hilbert space of a single D4 brane that correspond to a split attractor flow in supergravity. Moreover, these are precisely the contributions to the Euler character of the D4-brane moduli space which come from the corner where the brane wrapping ample divisor $D = ND_1 + MD_2$ fragments into N D4 branes on D_1 and M D4 branes on D_2 .

Consider a 4-brane wrapping a very amply decomposable divisor, $D = D_1 + D_2 \in H^2(X)$, in the Calabi-Yau. Lifting to M-theory, the partition function with chemical potentials turned on for the D2 and D0 fluxes is given by the MSW CFT that is essentially a

$(0, 4)$ sigma model into the moduli space of deformations of the 4-brane. The exact formulation of this conformal field theory is unknown, but the dominant classical contribution is believed to reduce to the Euler character of the nonsingular part of the classical moduli space of a surface in the class $[D]$.

We are interested in understanding a particular subleading correction to the partition function that comes from the singular corner of the 4-brane moduli space where the surface splits into a pair wrapping D_1 and D_2 and intersecting over the common curve, Σ_g . In particular, we argue that this contribution to the entropy is best analyzing using the Vafa-Witten worldvolume theory living on the 4-branes, and the topological bifundamental matter coupling them along the intersection. Furthermore, we speculate that this piece of the full partition function is associated to certain two centered black holes states in the supergravity limit. Therefore our partition function should be interpreted as the microscopic quantum theory describing the jump in the index of BPS states across the wall of marginal stability for the $D \rightarrow D_1 + D_2$ split, at least in the case of local Calabi-Yau.

In compact models [14] demonstrated that the marginal stability line for $D4 \rightarrow D4 + D4$ split can be crossed by starting a flow from large values of background Kähler modulus t_∞ . We now show that this is also the case in our non-compact model. Let $\Gamma_1(\Gamma_2)$ be the charge of N $D4$ -branes (M $D4$ -branes) wrapped on $D_1(D_2)$. Let ω_i be Poincare Dual of divisor D_i for $i = 1, 2$. Then the charges including fluxes on the branes are given by

$$\begin{aligned}\Gamma_1 &= N\omega_1 + \left(\frac{1}{2}N^2\omega_1^2 - f_1\omega_1\omega_2\right) + q_0dV \\ \Gamma_2 &= M\omega_2 + \left(\frac{1}{2}M^2\omega_2^2 - f_2\omega_1\omega_2\right) + q'_0dV,\end{aligned}$$

where dV is the volume form, and we use the same sign conventions as [14].

Let $B + iJ = T_1\omega_1 + T_2\omega_2$ be the complexified Kähler form. In our geometry we know intersection numbers

$$C_{112} = g - 1 - p, \quad C_{122} = g - 1 + p$$

Furthermore, the numbers C_{111} and C_{222} (which are a priori ambiguous due to non-compactness) can be extracted from [2] where classical contribution to free energy was fixed, i.e. we relate

$$\frac{T^3}{(g-1+p)(g-1-p)} = C_{111}T_1^3 + C_{122}T_1T_2^2 + C_{211}T_2T_1^2 + C_{222}T_2^3$$

where $T = T_1 C_{112} + T_2 C_{122}$ is the complexified Kähler modulus of Riemann surface Σ_g . This procedure gives

$$C_{111} = \frac{C_{112}^2}{C_{122}}, \quad C_{222} = \frac{C_{122}^2}{C_{112}}$$

Now we can compute intersection pairing for the charges which is generically non-zero for $f_1 \neq f_2$. For example, if $p = 0$, then

$$\langle \Gamma_1, \Gamma_2 \rangle = (g-1) \left(\frac{1}{2} N M^2 - N f_2 - \frac{1}{2} M N^2 + M f_1 \right) \neq 0.$$

Let us consider the region in the moduli space with $Im T_1 \gg 1$, $Im T_2 \gg 1$. Then, we evaluate central charges $Z_1 = Z(\Gamma_1)$ and $Z_2 = Z(\Gamma_2)$ as

$$Z_1 = -\frac{1}{2} \frac{N T^2}{C_{122}} + T \left(\frac{1}{2} \frac{N^2 C_{112}}{C_{122}} - f_1 \right) - q_0$$

$$Z_2 = -\frac{1}{2} \frac{M T^2}{C_{112}} + T \left(\frac{1}{2} \frac{M^2 C_{122}}{C_{112}} - f_2 \right) - q'_0$$

Please note that dependence on the complexified Kähler moduli T_1, T_2 in both Z_1 and Z_2 comes only via the complexified Kähler modulus T of Σ_g .

Let us write $T = x + iy$ and compute $Im(Z_1 \bar{Z}_2)$:

$$Im(Z_1 \bar{Z}_2) = y \left(\frac{\alpha}{2} y^2 + \beta(x) \right)$$

where

$$\alpha = \frac{1}{2} \frac{M N^2}{C_{122}} - \frac{1}{2} \frac{N M^2}{C_{112}} + \frac{N f_2}{C_{122}} - \frac{M f_1}{C_{112}}$$

$$\beta(x) = \alpha x^2 + x \left(\frac{N q'_0}{C_{122}} - \frac{M q_0}{C_{112}} \right) + q_0 \left(\frac{M^2 C_{122}}{2 C_{112}} - f_2 \right) - q'_0 \left(\frac{N^2 C_{112}}{2 C_{122}} - f_1 \right)$$

To find a solution (x_{MS}, y_{MS}) of $Im(Z_1 \bar{Z}_2) = 0$, with $y_{MS} \gg 1$ we need to have

$$\frac{\beta(x)}{\alpha} < 0. \quad \left| \frac{\beta(x)}{\alpha} \right| \gg 1$$

This can clearly be done for a choice of B-field and 2-form fluxes f_1, f_2 as well as a choice of flux-induced D0 charges q_0, q'_0 . For example, consider $N = M$ and $p = 0$. Let us further assume $x^2 \ll N$, choose $f_2 - f_1$ to be of order 1 and positive, $q_0 - q'_0$ to be of order 1 and negative and $q'_0 f_1 - q_0 f_2$ be of order much less than N^2 . With this choices, we find

$$\beta = \frac{N^2 (q_0 - q'_0)}{2} < 0$$

$$\alpha = \frac{N}{g-1}(f_2 - f_1) > 0$$

so that the ratio $\frac{\beta}{\alpha}$ is of order N and negative.

The above consideration demonstrates the existence of marginal stability wall for $D4 \rightarrow D4 + D4$ split only for large N, M , however we find it plausible that such a wall exists even for finite N, M . As an evidence that our results may be related with split flows of [13] even for finite N, M note the following. Let us consider $N = M = 1$ and $p = 0$. Then, inserting the factor Σ^{2g-2} in the measure of path-integral in eq.(2.5)(the result of integrating out bifundamentals) in the limit of small coupling g_s is equivalent to

$$g_s(g-1) \left(Z^{(M)} \frac{\partial}{\partial \theta} Z^{(N)} - Z^{(N)} \frac{\partial}{\partial \theta} Z^{(M)} \right). \quad (5.1)$$

But this is nothing else but inserting angular momentum degeneracy

$$\langle \Gamma_1, \Gamma_2 \rangle = (g-1)(NQ_2 - MQ_1)$$

as in [14], since the derivatives with respect to θ simply pull down factors of $\frac{1}{g_s} Q_{D2}$. It would be interesting to investigate this connection further. Below we propose a tentative interpretation of the factor Σ^{2g-2} (for $N = M = 1$ and $p = 0$) as arising from the halo of $D2/D0$ -branes surrounding the two $D4$ -fragments.

Consider $D2/D0$ system with charge and central charge given by

$$\Gamma_3 = -f_3 \omega_1 \omega_2 + q_3 [pt], \quad Z_3 = -f_3 T - q_3.$$

Let us find solutions of $Im(Z_1 \bar{Z}_3) = 0$, where

$$Im(Z_1 \bar{Z}_3) = y \left(\frac{\gamma}{2} y^2 + \delta(x) \right)$$

with

$$\gamma = \frac{Nf_3}{C_{122}}, \quad \delta(x) = \gamma x^2 + x \left(\frac{Nq_3}{C_{122}} \right) - q_0 f_3 - q_3 \left(\frac{N^2 C_{112}}{2C_{122}} - f_1 \right)$$

Let us look for solutions in the region $x^2 \ll N$. We may choose f_3 to be positive of order 1, $q_3 f_1 - q_0 f_3 \ll N^2$ and $q_3 > 0$. Then, the ration $\frac{\delta}{\gamma}$ is negative and of order N . Hence, there exists marginal stability wall for the charge Γ_1 and Γ_3 precisely in the same region- $x^2 \ll N$ and $y^2 \sim N$ -as MW for the Γ_1, Γ_2 charges found above. Again, this argument holds for large N, M but let us assume that the conclusion is also true for finite N, M , i.e.

there exists a halo of $D2/D0$ particles, which are mutually BPS with the two fragmented D4-branes. If each such particle has one unit of D2 charge wrapping Σ_g and $n - m$ units of D0 charge, then the angular momentum degeneracy is $\langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle = 2g - 2$. If each such particle is a fermion and u is chemical potential, then one finds the factor $(1 - u^{n-m})^{2g-2}$. Our factor Σ^{2g-2} is a modular transformed version of this.

There is a subtlety in the generating function for the index of split states in a non-compact geometry, which modifies (5.1). As explained by [12], the entropy computed by the q-deformed Yang-Mills is in the mixed ensemble for the local D0 and D2 charges, but the D2 charges associated to 2-cycles that become noncompact in the local limit are held fixed. Formally, one has the relation

$$Z_{local}(\phi_0, \phi; F) = \int du Z_{mixed}(\phi_0, \phi, u) e^{F \cdot u},$$

where $F \in H_2(X)$ such that $F \cdot [\Sigma_g] = 0$ is the fixed noncompact D2 flux, and u are the associated chemical potentials.

Therefore one expects that the local split partition function will be determined in terms of the fragments via

$$Z_{local}(\phi_0, \phi; F) = \sum_{f \in H_2} (N Z_{local}^1(\phi_0, \phi; f) D_1 \cdot (F - f - \partial) Z_{local}^2(\phi_0, \phi; f - F) \\ - M Z_{local}^2(\phi_0, \phi; f) D_2 \cdot (f - \partial) Z_{local}^1(\phi_0, \phi; f)),$$

where the sum is over $f \cdot [\Sigma_g] = 0$ that can be supported as flux on D_1 and $F - f$ can be obtained as flux on D_2 .

In the large charge limit, this is exactly the form that we find in the factorization of the coupled q-deformed Yang-Mills (4.3), where the “ghost representations” parametrize these noncompact flux sectors. The integrals over the noncompact moduli serve to enforce the condition that the noncompact D2 charge bound as flux in the two fragments cancel.

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